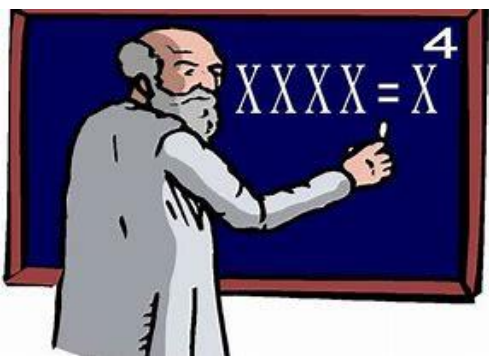

FRACTIONAL EXPONENTS

We know what n^5 means ($n \cdot n \cdot n \cdot n \cdot n$), and we might know what x^{-3} means ($1/x^3$). You should also commit to memory that $w^1 = w$ for any number w , and that $y^0 = 1$ for any number y (that isn't zero). But what could something like $a^{5/3}$ mean . . . an exponent that's a fraction?? Stick around and you'll find out.



□ REVIEW OF EXPONENT LAWS

Let's begin by reviewing The Five Laws of Exponents:



$$x^a x^b = x^{a+b}$$

$$\frac{x^a}{x^b} = x^{a-b}$$

$$(x^a)^b = x^{ab}$$

$$(xy)^a = x^a y^a$$

$$\left(\frac{x}{y}\right)^a = \frac{x^a}{y^a}$$

Homework

1. Simplify each expression (assumes you've studied *Negative Exponents*):

a. $x^3 x^7$	b. $\frac{a^{10}}{a^5}$	c. $(z^3)^5$	d. $\left(\frac{x}{y}\right)^9$
e. $(mn)^6$	f. $r^4 t^5$	g. $a^2 + a^4$	h. $g^6 + g^6$
i. $(x + y)^2$	j. $x^4 x^{-3}$	k. $(c^{-3})^{-5}$	l. $(ab)^{-4}$
m. $\frac{x^{-8}}{x^{-6}}$	n. $\frac{a^{-4}}{a^4}$	o. $\left(\frac{u}{w}\right)^{-3}$	p. $(a - b)^2$

□ THE MEANING OF A FRACTIONAL EXPONENT

Now for a new kind of exponent: fractional. For example, is there any reasonable meaning for the expression “9 to the $1/2$ power”: $9^{1/2}$? To determine the meaning of this number, we can give $9^{1/2}$ a name; call it R . Now check out the following:

$$\begin{aligned}
 R &= 9^{1/2} \\
 \Rightarrow R^2 &= (9^{1/2})^2 && \text{(square each side, just to see what happens)} \\
 \Rightarrow R^2 &= 9^{\frac{1}{2} \cdot 2} && \text{(law of exponents)} \\
 \Rightarrow R^2 &= 9^1 && \text{(1/2 and 2 are reciprocals)} \\
 \Rightarrow R^2 &= 9 \\
 \Rightarrow R &= \pm 3 && \text{(don't forget that there are 2 solutions)}
 \end{aligned}$$

Since R is equal to both $9^{1/2}$ and ± 3 , it follows that

$$9^{1/2} = \pm 3$$

But we can't have two answers to one question, so we'll keep the positive 3 and ignore the negative 3, and declare that

$$9^{1/2} = 3$$

What are we saying here? Since we know that $\sqrt{9} = 3$, we're saying that

$9^{1/2}$ simply means the **positive square root of 9**.

Without any more gory details, we could prove that all fractional exponents translate to radicals:

$$x^{1/2} = \sqrt{x} \quad x^{1/3} = \sqrt[3]{x} \quad x^{1/4} = \sqrt[4]{x}$$

Examples:

$$9^{1/2} = 3 \quad 100^{1/2} = 10 \quad 25^{1/2} = 5$$

$$64^{1/3} = 4 \quad 8^{1/3} = 2 \quad 125^{1/3} = 5$$

$$16^{1/4} = 2 \quad 81^{1/4} = 3 \quad 256^{1/4} = 4$$

Summary:

Assuming n is a natural number bigger than 1:

$$x^{1/n} = \sqrt[n]{x} \quad \text{for } n = 2, 3, 4, \dots$$

EXAMPLE 1: Evaluate each expression with a fractional exponent:

A. $225^{1/2} = \sqrt{225} = 15$

B. $125^{1/3} = \sqrt[3]{125} = 5$

C. $81^{1/4} = \sqrt[4]{81} = 3$

D. $(-32)^{1/5} = \sqrt[5]{-32} = -2$

E. $-16^{1/2} = -\sqrt{16} = -4$

F. $(-16)^{1/2} = \sqrt{-16} = \text{Not a real number}$

G. $-81^{1/4} = -\sqrt[4]{81} = -3$

H. $(-81)^{1/4} = \sqrt[4]{-81} = \text{Not a real number}$

Homework

2. Explain why $25^{1/2}$ is a real number, but $(-25)^{1/2}$ is not.

3. Evaluate each expression:

a. $36^{1/2}$

b. $8^{1/3}$

c. $16^{1/4}$

d. $32^{1/5}$

e. $625^{1/2}$

f. $1^{1/3}$

g. $0^{1/4}$

h. $243^{1/5}$

i. $-25^{1/2}$

j. $-49^{1/2}$

k. $(-64)^{1/2}$

l. $(-16)^{1/4}$

m. $(-64)^{1/3}$

n. $(-1)^{1/5}$

o. $(-32)^{1/5}$

p. $(-1)^{1/4}$

4. Convert each expression to radical form:

a. $x^{1/2}$

b. $y^{1/3}$

c. $z^{1/4}$

d. $w^{1/5}$

$$\begin{array}{llll}
 \text{e. } (ab)^{1/2} & \text{f. } ab^{1/2} & \text{g. } xy^{1/3} & \text{h. } (xy)^{1/3} \\
 \text{i. } y+z^{1/2} & \text{j. } (y+z)^{1/2} & \text{k. } (a-b)^{1/3} & \text{l. } (Q+R-T)^{1/4}
 \end{array}$$

5. Convert each expression to exponent form:

$$\begin{array}{llll}
 \text{a. } \sqrt{x} & \text{b. } \sqrt[4]{y} & \text{c. } \sqrt[3]{z} & \text{d. } \sqrt[5]{n} \\
 \text{e. } a\sqrt{b} & \text{f. } \sqrt{ab} & \text{g. } x\sqrt[4]{y} & \text{h. } \sqrt[3]{tw} \\
 \text{i. } \sqrt{x+y} & \text{j. } \sqrt[3]{p-q} & \text{k. } \sqrt[4]{a+n} & \text{l. } \sqrt[6]{x-x}
 \end{array}$$

6. True/False:

- The expression $x^{1/2}$ is always defined.
- The expression $x^{1/3}$ is always defined.

❏ **MORE FRACTIONAL EXPONENTS**

The previous problems each had a numerator of 1 in the fractional exponent. What about an expression like $27^{2/3}$? What could this mean? Let's dissect $27^{2/3}$ using our laws of exponents to determine the value of this number.

$$\begin{array}{ll}
 27^{2/3} & \text{(the power of 27 we're analyzing)} \\
 = 27^{\frac{1}{3} \cdot 2} & \text{(certainly } \frac{1}{3} \cdot 2 = \frac{2}{3} \text{)} \\
 = \left(27^{1/3}\right)^2 & \text{(law of exponents: } (x^a)^b = x^{ab} \text{)} \\
 = \left(\sqrt[3]{27}\right)^2 & \text{(a } 1/3 \text{ exponent indicates cube root)} \\
 = 3^2 & \text{(the cube root of 27 is 3)} \\
 = 9 & \text{(3 squared is 9)}
 \end{array}$$

Here's another example. Let's calculate $16^{5/4}$:

$$16^{5/4} = 16^{\frac{1}{4} \cdot 5} = \left(16^{1/4}\right)^5 = \left(\sqrt[4]{16}\right)^5 = 2^5 = \mathbf{32}$$

And a third example:

$$243^{2/5} = 243^{\frac{1}{5} \cdot 2} = \left(243^{1/5}\right)^2 = \left(\sqrt[5]{243}\right)^2 = 3^2 = \mathbf{9}$$

Homework

7. Using the three examples above as a guide, find the value of each fractional power:

- | | | | |
|---------------|---------------|---------------|---------------|
| a. $8^{2/3}$ | b. $4^{3/2}$ | c. $9^{3/2}$ | d. $16^{5/2}$ |
| e. $27^{4/3}$ | f. $27^{2/3}$ | g. $16^{3/4}$ | h. $32^{7/5}$ |

❑ ***THE BOTTOM LINE FOR FRACTIONAL EXPONENTS***

What's really going on here? There must be a simpler way to view a fractional exponent, and thus a simpler way to calculate one. If you look at the above examples and the homework you just completed, you may have noticed that the denominator of the fractional exponent indicated a root, while the numerator indicated a power. For example, in the first example above, we showed that $27^{2/3}$ was calculated by first taking the cube root of 27, which is 3, and then squaring that result to get **9**.

Thus, a problem like $16^{5/2}$ is calculated quickly as the square root of 16, raised to the fifth power, which is 4^5 , which is **1,024**.

In summary,

$$x^{p/q} = \left(\sqrt[q]{x} \right)^p$$

$x^{\frac{p}{q}}$

Power Root

EXAMPLE:

$$\begin{aligned}
 & 8^{5/3} \\
 &= \left(\sqrt[3]{8} \right)^5 \\
 &= 2^5 \\
 &= 32
 \end{aligned}$$

One more example for this section: To calculate $64^{2/3}$, think cube root of 64, raised to the second power, which is 4^2 , which is **16**.

Homework

8. Using the power-root idea, find the value of each fractional power:

a. $8^{4/3}$ b. $4^{1/2}$ c. $9^{5/2}$ d. $16^{3/2}$

e. $27^{2/3}$ f. $27^{4/3}$ g. $16^{5/4}$ h. $32^{6/5}$

i. $8^{2/3}$ j. $4^{3/2}$ k. $9^{3/2}$ l. $16^{5/2}$

m. $27^{4/3}$ n. $125^{2/3}$ o. $16^{3/4}$ p. $32^{7/5}$

❑ **NEGATIVE FRACTIONAL EXPONENTS**

Since a negative exponent indicates a reciprocal, we can combine negative exponents (assuming you've studied them) with the fractional exponents we're learning now. Recalling that $x^{-n} = \frac{1}{x^n}$, we can work the following examples.

EXAMPLE 2: Evaluate each expression:

$$\text{A. } 9^{-1/2} = \frac{1}{9^{1/2}} = \frac{1}{\sqrt{9}} = \frac{1}{3}$$

$$\text{B. } -27^{-1/3} = -\frac{1}{27^{1/3}} = -\frac{1}{\sqrt[3]{27}} = -\frac{1}{3}$$

$$\text{C. } 8^{-2/3} = \frac{1}{8^{2/3}} = \frac{1}{(\sqrt[3]{8})^2} = \frac{1}{2^2} = \frac{1}{4}$$

$$\text{D. } 9^{-5/2} = \frac{1}{9^{5/2}} = \frac{1}{(\sqrt{9})^5} = \frac{1}{3^5} = \frac{1}{243}$$

$$\text{E. } -16^{-3/4} = -\frac{1}{16^{3/4}} = -\frac{1}{(\sqrt[4]{16})^3} = -\frac{1}{2^3} = -\frac{1}{8}$$

$$\text{F. } (-16)^{-3/2} = \frac{1}{(-16)^{3/2}} = \frac{1}{(\sqrt{-16})^3} = \text{Not a real number}$$

$$\text{G. } (-8)^{-2/3} = \frac{1}{(-8)^{2/3}} = \frac{1}{(\sqrt[3]{-8})^2} = \frac{1}{(-2)^2} = \frac{1}{4}$$

Homework

9. Evaluate each expression:

- a. $9^{-3/2}$ b. $27^{-2/3}$ c. $8^{-4/3}$ d. $8^{-1/3}$
 e. $16^{-5/4}$ f. $81^{-1/4}$ g. $81^{-3/4}$ h. $32^{-1/5}$
 i. $-8^{-4/3}$ j. $(-25)^{-3/2}$ k. $(-64)^{-2/3}$ l. $(-32)^{-6/5}$

10. Convert each expression to radical form:

- a. $x^{2/3}$ b. $y^{5/4}$ c. $z^{1/7}$
 d. $(a + b)^{3/2}$ e. $(x - y)^{4/3}$ f. $(xy)^{4/5}$
 g. $uw^{2/3}$ h. $x + y^{4/5}$ i. $(ab + c)^{5/6}$

11. Convert each expression to exponent form:

- a. $\sqrt[3]{t}$ b. $\sqrt[5]{z}$ c. $\sqrt[4]{ab}$
 d. $x\sqrt[3]{y}$ e. $(\sqrt{a})^3$ f. $(\sqrt[3]{p})^2$
 g. $(\sqrt[4]{w})^5$ h. $(\sqrt[7]{a+b})^3$ i. $(\sqrt{x-y})^{10}$

❑ THE LAWS OF EXPONENTS

The same five laws of exponents we've used with all the previous exponents still work just fine with fractional exponents.

EXAMPLE 3: Simplify each expression:

$$\begin{aligned} \text{A. } a^{1/2}a^{1/3} &= a^{1/2+1/3} = a^{3/6+2/6} = a^{5/6} \\ &\text{(add the exponents)} \end{aligned}$$

B. $y^{1/2}y^{1/2} = y^{1/2+1/2} = y^1 = y$
(add the exponents)

C. $(x^{3/4})^{7/2} = x^{(3/4)(7/2)} = x^{21/8}$
(multiply the exponents)

D. $\frac{n^{5/6}}{n^{2/3}} = n^{5/6-2/3} = n^{5/6-4/6} = n^{1/6}$
(subtract the exponents)

E. $(ab)^{4/5} = a^{4/5}b^{4/5}$
(raise each factor to the 4/5 power)

F. $\left(\frac{a}{b}\right)^{5/3} = \frac{a^{5/3}}{b^{5/3}}$
(raise top and bottom to the 5/3 power)

G. $\frac{w^{1/2}}{w^{4/5}} = w^{1/2-4/5} = w^{5/10-8/10} = w^{-3/10} = \frac{1}{w^{3/10}}$
(subtract the exponents) (LCD) (a negative exponent means reciprocal)

Homework

12. Simplify each expression:

a. $x^{1/2}x^{2/3}$	b. $\frac{y^{4/5}}{y^{1/5}}$	c. $\frac{w^{1/2}}{w^{2/3}}$	d. $(a^{2/7})^7$
e. $\left(\frac{p}{q}\right)^{3/8}$	f. $t^{4/5}t^{1/3}$	g. $(k^{5/2})^{2/5}$	h. $(wz)^{2/3}$

EXAMPLE 4: Simplify each expression:

$$\text{A. } x^{-1/2} x^{-2/3} = x^{-1/2 - 2/3} = x^{-3/6 - 4/6} = x^{-7/6} = \frac{1}{x^{7/6}}$$

$$\text{B. } \frac{n^{1/2}}{n^{-4/5}} = n^{1/2 - (-4/5)} = n^{1/2 + 4/5} = n^{5/10 + 8/10} = n^{13/10}$$

$$\text{C. } \left(x^{2/3}\right)^{-1/4} = x^{(2/3)(-1/4)} = x^{-1/6} = \frac{1}{x^{1/6}}$$

$$\text{D. } (xy)^{-2/3} = x^{-2/3} y^{-2/3} = \frac{1}{x^{2/3}} \cdot \frac{1}{y^{2/3}} = \frac{1}{x^{2/3} y^{2/3}}$$

$$\text{E. } \left(\frac{g}{h}\right)^{-4/3} = \frac{g^{-4/3}}{h^{-4/3}} = \frac{\frac{1}{g^{4/3}}}{\frac{1}{h^{4/3}}} = \frac{1}{g^{4/3}} \times \frac{h^{4/3}}{1} = \frac{h^{4/3}}{g^{4/3}}$$

Homework

13. Simplify each expression:

a. $x^{4/5} x^{-3/5}$

b. $y^{1/3} y^{-5/3}$

c. $\left(a^{-1/2}\right)^{-2/3}$

d. $(abc)^{-3/4}$

e. $\left(\frac{w}{z}\right)^{-2/5}$

f. $\frac{n^{-1/2}}{n^{-2/3}}$

g. $\frac{a^{-3}}{a^{5/2}}$

h. $\frac{x}{x^{-2/3}}$

i. $\left(\left(c^{1/2}\right)^{-4/3}\right)^{-3/2}$

Review Problems

14. Convert $\sqrt[4]{x+y}$ to exponent form.
15. Convert $ab^{2/3}$ to radical form.
16. Convert $(a+b)^{5/2}$ to radical form.
17. Convert $\sqrt{a^3-x^3}$ to exponent form.
18. Evaluate: a. $9^{1/2}$ b. $64^{1/3}$ c. $81^{1/4}$ d. $32^{1/5}$
19. Evaluate: a. $8^{2/3}$ b. $27^{4/3}$ c. $32^{2/5}$ d. $16^{3/4}$
20. Evaluate: a. $-9^{1/2}$ b. $(-9)^{1/2}$ c. $(-8)^{1/3}$ d. $(-16)^{1/4}$
21. Evaluate: a. $9^{-3/2}$ b. $8^{-4/3}$ c. $125^{-4/3}$ d. $-81^{-1/4}$
22. Simplify: a. $x^{1/2}x^{4/5}$ b. $\frac{a^{1/3}}{a^{2/5}}$ c. $(ab)^{4/7}$
23. Simplify: a. $\left(\frac{a}{b}\right)^{2/7}$ b. $\left(x^{2/3}\right)^{3/5}$ c. $a^{1/2}a^{1/3}a^{1/4}$
24. Simplify: a. $y^{1/2}y^{-1/2}$ b. $\frac{n^{1/3}}{n^{-4/3}}$ c. $(PQ)^{-2/3}$
25. Simplify: a. $\left(\frac{x}{w}\right)^{-7/10}$ b. $\left(p^{-2/3}\right)^{5/6}$ c. $x^{2/3}+x^{1/3}$

Solutions

1. a. x^{10} b. a^5 c. z^{15} d. $\frac{x^9}{y^9}$ e. m^6n^6
 f. As is g. As is h. $2g^6$ i. $x^2 + 2xy + y^2$
 j. x k. c^{15} l. $\frac{1}{a^4b^4}$ m. $\frac{1}{x^2}$ n. $\frac{1}{a^8}$
 o. $\frac{w^3}{u^3}$ p. Assuming you've learned how to multiply binomials, the answer is $a^2 - 2ab + b^2$
2. $25^{1/2} = \sqrt{25} = 5$, a real number. But $(-25)^{1/2} = \sqrt{-25}$, not a real number.
3. a. 6 b. 2 c. 2 d. 2 e. 25 f. 1 g. 0 h. 3 i. -5
 j. -7 k. Not real l. Not real m. -4 n. -1 o. -2
 p. Not real
4. a. $\sqrt{x}$ b. $\sqrt[3]{y}$ c. $\sqrt[4]{z}$ d. $\sqrt[5]{w}$ e. $\sqrt{ab}$
 f. $a\sqrt{b}$ g. $x\sqrt[3]{y}$ h. $\sqrt[3]{xy}$ i. $y + \sqrt{z}$ j. $\sqrt{y+z}$
 k. $\sqrt[3]{a-b}$ l. $\sqrt[4]{Q+R-T}$
5. a. $x^{1/2}$ b. $y^{1/4}$ c. $z^{1/3}$ d. $n^{1/5}$ e. $ab^{1/2}$ f. $(ab)^{1/2}$
 g. $xy^{1/4}$ h. $(tw)^{1/3}$ i. $(x+y)^{1/2}$ j. $(p-q)^{1/3}$ k. $(a+n)^{1/4}$ l. 0
6. a. False; if $x = -9$, for instance, then $(-9)^{1/2} = \sqrt{-9}$ which is not a real number, which means that $x^{1/2}$ is undefined in this class when $x = -9$. In fact, $x^{1/2}$ is undefined whenever x is a negative number.
 b. True; since $x^{1/3} = \sqrt[3]{x}$, and since the cube root is defined whether x is positive, zero, or negative, $x^{1/3}$ is always defined.

7. a. $8^{2/3} = 8^{\frac{1}{3} \cdot 2} = \left(8^{1/3}\right)^2 = \left(\sqrt[3]{8}\right)^2 = 2^2 = 4$
 b. 8 c. 27 d. 1024 e. 81 f. 9 g. 8
 h. $32^{7/5} = 32^{\frac{1}{5} \cdot 7} = \left(32^{1/5}\right)^7 = \left(\sqrt[5]{32}\right)^7 = 2^7 = 128$
8. a. $8^{4/3}$ is the cube root of 8, raised to the 4th power: $2^4 = 16$
 b. 2
 c. $9^{5/2}$ is the square root of 9, raised to the 5th power: $3^5 = 243$
 d. 64 e. 9 f. 81 g. 32
 h. $32^{6/5}$ is the fifth root of 32, raised to the 6th power: $2^6 = 64$
 i. 4 j. 8 k. 27 l. 1024 m. 81 n. 25 o. 8 p. 128
9. a. $\frac{1}{27}$ b. $\frac{1}{9}$ c. $\frac{1}{16}$ d. $\frac{1}{2}$ e. $\frac{1}{32}$ f. $\frac{1}{3}$
 g. $\frac{1}{27}$ h. $\frac{1}{2}$ i. $-\frac{1}{16}$ j. Not real k. $\frac{1}{16}$ l. $\frac{1}{64}$
10. a. $\left(\sqrt[3]{x}\right)^2$ b. $\left(\sqrt[4]{y}\right)^5$ c. $\sqrt[7]{z}$ d. $\left(\sqrt{a+b}\right)^3$ e. $\left(\sqrt[3]{x-y}\right)^4$
 f. $\left(\sqrt[5]{xy}\right)^4$ g. $u\left(\sqrt[3]{w}\right)^2$ h. $x + \sqrt[5]{y^4}$ i. $\left(\sqrt[6]{ab+c}\right)^5$
11. a. $t^{1/3}$ b. $z^{1/5}$ c. $(ab)^{1/4}$ d. $xy^{1/3}$ e. $a^{3/2}$
 f. $p^{2/3}$ g. $w^{5/4}$ h. $(a+b)^{3/7}$ i. $(x-y)^{10/2} = (x-y)^5$
12. a. $x^{7/6}$ b. $y^{3/5}$ c. $\frac{1}{w^{1/6}}$ d. a^2
 e. $\frac{p^{3/8}}{q^{3/8}}$ f. $t^{17/15}$ g. k h. $w^{2/3}z^{2/3}$
13. a. $x^{1/5}$ b. $\frac{1}{y^{4/3}}$ c. $a^{1/3}$ d. $\frac{1}{a^{3/4}b^{3/4}c^{3/4}}$
 e. $\frac{z^{2/5}}{w^{2/5}}$ f. $n^{1/6}$ g. $\frac{1}{a^{11/2}}$ h. $x^{5/3}$
 i. c

14. $(x + y)^{1/4}$ 15. $a(\sqrt[3]{b})^2$
16. $(\sqrt{a+b})^5$ or $\sqrt{(a+b)^5}$
17. $(a^3 - x^3)^{1/2}$
18. a. 3 b. 4 c. 3 d. 2
19. a. 4 b. 81 c. 4 d. 8
20. a. -3 b. Not real c. -2 d. Not real
21. a. $\frac{1}{27}$ b. $\frac{1}{16}$ c. $\frac{1}{625}$ d. $-\frac{1}{3}$
22. a. $x^{13/10}$ b. $\frac{1}{a^{1/15}}$ c. $a^{4/7}b^{4/7}$
23. a. $\frac{a^{2/7}}{b^{2/7}}$ b. $x^{2/5}$ c. $a^{13/12}$
24. a. 1 b. $n^{5/3}$ c. $\frac{1}{P^{2/3}Q^{2/3}}$
25. a. $\frac{w^{7/10}}{x^{7/10}}$ b. $\frac{1}{p^{5/9}}$ c. As is

"If you limit your choices only to what seems possible or reasonable, you disconnect yourself from what you truly want, and all that is left is a compromise."

- Robert Fritz